

Computer-Aided Discovery and Categorisation of Personality Axioms

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Abstract

We propose a computer-algebraic, order-theoretic framework based on intuitionistic logic for the computer-aided discovery of personality axioms from personality-test data and their mathematical categorisation into formal personality theories in the spirit of F. Klein’s *Erlanger Programm* for geometrical theories. As a result, formal personality theories can be automatically generated, diagrammatically visualised, and mathematically characterised in terms of categories of invariant-preserving transformations in the sense of Klein and category theory. Our personality theories and categories are induced by implicational invariants that are ground instances of intuitionistic implication, which we postulate as axioms. In our mindset, the essence of personality, and thus mental health and illness, is its invariance. The truth of these axioms is algorithmically extracted from histories of partially-ordered, symbolic data of observed behaviour. The personality-test data and the personality theories are related by a Galois-connection in our framework. As data format, we adopt the format of the symbolic values generated by the Szondi-test, a personality test based on L. Szondi’s unifying, depth-psychological theory of fate analysis.

Keywords: applied order theory, computational and mathematical depth psychology, data mining, diagrammatic reasoning, fuzzy implication, intuitionistic logic, logical and visual data analytics, personality tests, Szondi.

1 Introduction

In 1872, Felix Klein, full professor of mathematics at the University of Erlangen at age 23, presented his influential *Erlanger Programm* [11, 12] on the classification and characterisation of geometrical theories by means of group theory. That is, Klein put forward the thesis that every geometrical theory could be characterised by an associated group of geometrical transformations that would leave invariant the essential properties of the geometrical objects of that theory. These essential properties are captured by the axioms that define the theory. As a result, geometrical theories could be classified in terms of their associated

transformation groups. According to [9], Klein’s *Erlanger Programm* “is regarded as one of the most influential works in the history of geometry, and more generally mathematics, during the half-century after its publication in 1872.”¹

In this paper and in the spirit of the *Erlanger Programm* for geometrical theories, we propose a computer-algebraic,² order-theoretic framework based on intuitionistic logic [17] for the computer-aided discovery of personality axioms from personality-test data and their mathematical categorisation into formal personality theories. Each one of the resulting intuitionistic personality theories is an (order-theoretic) *prime filter* [4] in our framework. As our contribution, formal personality theories can be automatically generated, diagrammatically visualised, and mathematically characterised in terms of categories of invariant-preserving transformations in the sense of Klein and category theory [16]. That is, inspired by and in analogy with Klein, we put forward the thesis that every personality theory can be characterised by an associated category of personality transformations that leave invariant the essential properties of “the personality objects”—the people, represented by their personality-test data—of that theory.

An important difference in our psychological context of personality theories to Klein’s geometrical context is that actually no formal personality theory in the strict axiomatic sense exists, whereas Klein could characterise a variety of *existing*, axiomatic theories of geometry. Before being able to categorise personality theories, we thus must first formally *define* them. As said, we shall do so by discovering their defining axioms from personality-test data with the aid of computers. Our personality theories and categories are then automatically induced by implicational invariants that hold throughout that test data and that are ground instances of intuitionistic implication, which we postulate as axioms. In our mindset, the essence of personality, and thus mental health and illness, is its invariance. So for every person, represented by her personality-test result P —the data—we automatically generate her associated

1. personality *theory* $\{P\}^\triangleleft$ of simple implicational invariants and
2. personality *category* $\mathbf{T}_{\{P\}^\triangleleft}$ of theory-preserving transformations.

(We are actually able to carry out this construction for whole *sets* of personality-test results, either of different people or of one and the same person.) More precisely, the truth of these axioms is algorithmically extracted from histories of partially-ordered, symbolic data of the person’s observed test behaviour. Our axioms have an implicational form in order to conform with the standard of Hilbert-style axiomatisations [7], which in our order-theoretic framework can be cast as a simple closure operator. Another difference in our context is that contrary to Klein, who worked with transformation groups, we work with more general transformation *monoids*, and thus transformation categories. The reason is that contrary to Klein’s geometrical context, in which transformations are invertible, transformations in the psychological context need not be invertible.

¹We add that in physics, the existence of certain transformation groups for mechanics and electromagnetism led Albert Einstein to discover his theory of special relativity.

²in the sense of *symbolic* as opposed to numeric computation

In our order-theoretic framework, personality-test data and personality theories are related by a Galois-connection $(\triangleright, \triangleleft)$ [4, Chapter 7]. As data format, we adopt—without loss of generality—the format of the symbolic values, called *Szondi personality profiles (SPPs)*, generated by the Szondi-test [21], a personality test based on L. Szondi’s unifying, depth-psychological theory of fate analysis [22]. An SPP can be conceived as a tuple of eight, so-called *signed factors* whose signatures can in turn take twelve values. We stress that our framework is independent of any personality test. It simply operates on the result values that such tests generate. Our choice of the result values of the Szondi-test is motivated by the fact that SPPs just happen to have a finer structure than other personality-test values that we are aware of, and so are perhaps best suited to play the illustrative role for which we have chosen them here. (See also [13].)

The remaining part of this paper is structured as follows: in Section 2, we present the part of our framework for the computer-aided discovery of personality axioms from personality-test data, and in Section 3, the part for their mathematical categorisation into formal personality theories.

2 Axiom discovery

In this section, we present the part of our framework for the computer-aided discovery of personality axioms from personality-test data. This is the data-mining and the logical and visual data-analytics part of our contribution.

We start with defining the format of the data on which we perform our data-mining and data-analytics operations. As announced, it is the format of the symbolic values, called *Szondi personality profiles (SPPs)*, generated by the Szondi-test [21]. We operate on finite sequences thereof. In diagnostic practice, these test-result sequences are usually composed of 10 SPPs [14].

Definition 1 (The Szondi-Test Result Space). Let us consider the Hasse-diagram [4] in Figure 1 of the partially ordered set of *Szondi’s twelve signatures* [21] of human reactions, which are:

- approval: from strong +!!!, +!! , and +! to weak + ;
- indifference/neutrality: 0 ;
- rejection: from weak − , −! , and −!! to strong −!!! ; and
- ambivalence: $\pm^!$ (approval bias), \pm (no bias), and $\pm_!$ (rejection bias).

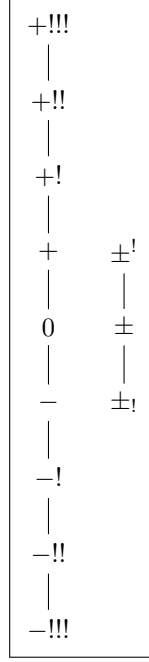
(Szondi calls the exclamation marks in his signatures *quanta*.)

Further let us call this set of signatures \mathbb{S} , that is,

$$\mathbb{S} := \{ -!!!, -!!, -!, -, 0, +, +!, +!!, +!!!, \pm_!, \pm, \pm^! \}.$$

Now let us consider *Szondi’s eight factors and four vectors* of human personality [21] as summarised in Table 1. (Their names are of clinical origin and

Figure 1: Hasse-diagram of Szondi's signatures



need not concern us here.) And let us call the set of factors \mathbb{F} , that is,

$$\mathbb{F} := \{ \mathbf{h}, \mathbf{s}, \mathbf{e}, \mathbf{hy}, \mathbf{k}, \mathbf{p}, \mathbf{d}, \mathbf{m} \}.$$

Then,

- $\text{SPP} := \{ ((\mathbf{h}, s_1), (\mathbf{s}, s_2), (\mathbf{e}, s_3), (\mathbf{hy}, s_4), (\mathbf{k}, s_5), (\mathbf{p}, s_6), (\mathbf{d}, s_7), (\mathbf{m}, s_8)) \mid s_1, \dots, s_8 \in \mathbb{S} \}$ is the set of Szondi's personality profiles;
- $\langle \text{SPP}^+, \star \rangle$ is the free semigroup on the set SPP^+ of all finite sequences of SPPs with \star the (associative) concatenation operation on SPP^+ and

$$\begin{aligned} \text{SPP}^+ &:= \bigcup_{n \in \mathbb{N} \setminus \{0\}} \text{SPP}^n \\ \text{SPP}^{1+n} &:= \text{SPP}^1 \times \text{SPP}^n \\ \text{SPP}^1 &:= \text{SPP}; \end{aligned}$$

- $\text{STR} := \langle \text{SPP}^+, \sqsubseteq \rangle$ is our *Szondi-Test Result Space*, where the suffix partial order \sqsubseteq on SPP^+ is defined such that for every $P, P' \in \text{SPP}^+$, $P \sqsubseteq P'$ if and only if $P = P'$ or there is $P'' \in \text{SPP}^+$ such that $P = P'' \star P'$.

Table 1: Szondi’s factors and vectors

| Vector | Factor | Signature | |
|------------------|---------------|----------------------|-----------------------|
| | | + | − |
| S (Id) | h (love) | physical love | platonic love |
| | s (attitude) | (proactive) activity | (receptive) passivity |
| P (Super-Ego) | e (ethics) | ethical behaviour | unethical behaviour |
| | hy (morality) | immoral behaviour | moral behaviour |
| Sch (Ego) | k (having) | having more | having less |
| | p (being) | being more | being less |
| C (Id) | d (relations) | unfaithfulness | faithfulness |
| | m (bindings) | dependence | independence |

As an example of an SPP, consider the *norm profile* for the Szondi-test [21]:

$$((h, +), (s, +), (e, -), (hy, -), (k, -), (p, -), (d, +), (m, +))$$

Spelled out, the norm profile describes the personality of a human being who approves of physical love, has a proactive attitude, has unethical but moral behaviour, wants to have and be less, and is unfaithful and dependent.

Those SPP-sequences that have been generated by a Szondi-test(ee) are our histories of partially-ordered, symbolic data of observed behaviour that we announced in the introduction. Table 2 displays an example of such an SPP-sequence: it is the so-called foreground profile of a 49-year old, male physician and psycho-hygienist and is composed of 10 subsequent SPPs [20, Page 182–184].

Fact 1 (Prefix closure of \sqsubseteq). For every $P, P', P'' \in \text{SPP}^+$,

$$P \sqsubseteq P' \text{ implies } P'' \star P \sqsubseteq P'$$

Proof. By inspection of definitions. \square

We continue to define the closure operator by which we generate our intuitionistic personality theories from personality-test data in the previously-defined format. Our personality theories are intuitionistic, because such theories can be interpreted over partially-ordered state spaces—such as our \mathcal{STR} —such that a sentence is true in the current state by definition if and only if the sentence is true in all states that are accessible from the current state by means of the partial order [15, 17]. In other words, the truth of such sentences is forward-invariant, which is precisely the property of sentences that we are looking for.

Definition 2 (A closure operator for intuitionistic theories). Let

$$\mathbb{A} := \{hs_1, ss_2, es_3, hys_4, ks_5, ps_6, ds_7, ms_8 \mid s_1, \dots, s_8 \in \mathcal{S}\}$$

be our set of atomic statements, and

$$\mathcal{L}(\mathbb{A}) \ni \phi ::= A \mid \phi \wedge \phi \mid \phi \vee \phi \mid \neg \phi \mid \phi \rightarrow \phi \quad \text{for } A \in \mathbb{A}$$

Table 2: A Szondi-test result (say P)

| Nr. | S | | P | | Sch | | C | |
|-----|---|---|---|----|-----|---|---|---|
| | h | s | e | hy | k | p | d | m |
| 1 | — | 0 | ± | ± | ± | ± | 0 | + |
| 2 | — | 0 | + | ± | ± | + | 0 | + |
| 3 | — | — | ± | ± | ± | + | + | ± |
| 4 | — | — | ± | + | + | + | 0 | + |
| 5 | — | 0 | 0 | + | ± | ± | 0 | + |
| 6 | — | 0 | ± | ± | ± | ± | + | ± |
| 7 | — | 0 | ± | ± | ± | + | 0 | + |
| 8 | — | — | 0 | ± | + | + | + | ± |
| 9 | — | 0 | ± | ± | ± | ± | 0 | + |
| 10 | — | 0 | 0 | ± | ± | + | 0 | + |

our logical language over \mathbb{A} , that is, the set of statements ϕ constructed from the atomic statements A and the intuitionistic logical connectives \wedge (conjunction, pronounced “and”), \vee (disjunction, pronounced “or”), \neg (negation, pronounced “henceforth not”), and \rightarrow (implication, pronounced “whenever—then”). As usual, we can macro-define falsehood as $\perp := A \wedge \neg A$ and truth as $\top := \neg \perp$.

Further let

$$\Gamma_0 := \{$$

- $\phi \rightarrow (\phi' \rightarrow \phi)$
- $(\phi \rightarrow (\phi' \rightarrow \phi'')) \rightarrow ((\phi \rightarrow \phi') \rightarrow (\phi \rightarrow \phi''))$
- $(\phi \wedge \phi') \rightarrow \phi$
- $(\phi \wedge \phi') \rightarrow \phi'$
- $\phi \rightarrow (\phi' \rightarrow (\phi \wedge \phi'))$
- $\phi \rightarrow (\phi \vee \phi')$
- $\phi' \rightarrow (\phi \vee \phi')$
- $(\phi \rightarrow \phi') \rightarrow ((\phi'' \rightarrow \phi') \rightarrow ((\phi \vee \phi'') \rightarrow \phi'))$
- $\perp \rightarrow \phi$

be our (standard) set of intuitionistic *axiom schemas*.

Then, $\text{Cl}(\emptyset) := \bigcup_{n \in \mathbb{N}} \text{Cl}^n(\emptyset)$, where for every $\Gamma \subseteq \mathcal{L}(\mathbb{A})$:

$$\begin{aligned} \text{Cl}^0(\Gamma) &:= \Gamma_0 \cup \Gamma \\ \text{Cl}^{n+1}(\Gamma) &:= \text{Cl}^n(\Gamma) \cup \\ &\quad \{ \phi' \mid \{ \phi, \phi \rightarrow \phi' \} \subseteq \text{Cl}^n(\Gamma) \} \quad (\text{modus ponens, MP}) \end{aligned}$$

We call $\text{Cl}(\emptyset)$ our *base theory*, and $\text{Cl}(\Gamma)$ a Γ -*theory* for any $\Gamma \subseteq \mathcal{L}(\mathbb{A})$.

The following standard fact asserts that we have indeed defined a closure operator. We merely state it as a reminder, because we shall use it in later proof developments. The term 2_{finite}^Γ denotes the set of all finite subsets of the set Γ .

Fact 2. The mapping $\text{Cl} : 2^{\mathcal{L}(\mathbb{A})} \rightarrow 2^{\mathcal{L}(\mathbb{A})}$ is a *standard consequence operator*, that is, a *substitution-invariant compact closure operator*:

1. $\Gamma \subseteq \text{Cl}(\Gamma)$ (extensivity)
2. if $\Gamma \subseteq \Gamma'$ then $\text{Cl}(\Gamma) \subseteq \text{Cl}(\Gamma')$ (monotonicity)
3. $\text{Cl}(\text{Cl}(\Gamma)) \subseteq \text{Cl}(\Gamma)$ (idempotency)
4. $\text{Cl}(\Gamma) = \bigcup_{\Gamma' \in 2_{\text{finite}}^\Gamma} \text{Cl}(\Gamma')$ (compactness)
5. $\sigma[\text{Cl}(\Gamma)] \subseteq \text{Cl}(\sigma[\Gamma])$ (substitution invariance),

where σ designates an arbitrary propositional $\mathcal{L}(\mathbb{A})$ -substitution.

Proof. For (1) to (4), inspect the inductive definition of Cl . And (5) follows from our definitional use of axiom *schemas*.³ \square

Note that in the sequel, “:iff” abbreviates “by definition, if and only if,” and

$$\begin{aligned} \Phi \vdash_\Gamma \phi & \text{ :iff } \Phi \subseteq \text{Cl}(\Gamma) \text{ implies } \phi \in \text{Cl}(\Gamma) \\ \vdash_\Gamma \phi & \text{ :iff } \emptyset \vdash_\Gamma \phi. \end{aligned}$$

We continue to define what we mean by our simple implicational invariants announced in the introduction. As announced there, these invariants are ground instances of intuitionistic implication, by which we mean that they are of the visually tractable, diagrammatic form $A \rightarrow A'$ rather than being of the more general, not generally visually tractable form $\phi \rightarrow \phi'$. As an example of what we mean by visually tractable, diagrammatic form, consider Table 3. For a given SPP-sequence P , we postulate the algorithmically extracted set $\mathcal{I}(P)$ of these invariants that hold throughout P (see Definition 3) as the axioms of the personality theory $\text{Cl}(\mathcal{I}(P))$ that we associate with P . These axioms thus capture those *logical dependencies* between signed factors that are invariant in P in the sense of holding throughout P . The algorithm for this axiom extraction and visualisation is displayed in Listing 1 and will be explained shortly. Note that given that these invariants hold throughout a sequence that has been generated by an iterated procedure, that is, an iterated execution of the Szondi-test, they can also be understood as *loop invariants*, which is a core concept in the science of computer programming [6]. So our algorithm for finding psychological invariants can actually also be understood and even be used as a method for inferring loop invariants from program execution traces in computer science.

³Alternatively to axiom schemas, we could have used axioms together with an additional substitution-rule set $\{\sigma[\phi] \mid \phi \in \text{Cl}^n(\Gamma)\}$ in the definiens of $\text{Cl}^{n+1}(\Gamma)$.

Table 3: The diagram of $\mathcal{I}(P)$ as extracted from the P in Table 2

| | h | s | e | hy | k | p | d | m |
|----|---|---|---|----|---|---|---|---|
| h | | | | | | | | |
| s | | | | | | | | |
| e | | | | | | | | |
| hy | | | | | | | | |
| k | | | | | | | | |
| p | | | | | | | | |
| d | | | | | | | | |
| m | | | | | | | | |

Definition 3 (Simple implicational invariants). Let the mapping $p : \text{SPP} \rightarrow \mathcal{L}(\mathbb{A})$ be such that

$$p(((h, s_1), (s, s_2), (e, s_3), (hy, s_4), (k, s_5), (p, s_6), (d, s_7), (m, s_8))) = hs_1 \wedge ss_2 \wedge es_3 \wedge hys_4 \wedge ks_5 \wedge ps_6 \wedge ds_7 \wedge ms_8.$$

Then, define the mapping $\mathcal{I} : \text{SPP}^+ \rightarrow 2^{\mathcal{L}(\mathbb{A})}$ of *simple implicational invariants* such that for every $P \in \text{SPP}^+$,

$$\mathcal{I}(P) := \{ A \rightarrow A' \mid \text{for every } P' \in \text{SPP}^+, \text{ if } P \sqsubseteq P' \text{ then } p(\pi_1(P')) \vdash_{\emptyset} A \rightarrow A' \},$$

where $\pi_1 : \text{SPP}^+ \rightarrow \text{SPP}$ is projection onto the first SPP component.

Notice the three implications “if—then,” \vdash , and \rightarrow of different logical level, and note that we use “if—then” and “implies” synonymously. This definition can be cast into an algorithm of linear complexity in the length of P , for example as described by the Java-program displayed in Listing 1, and the result $\mathcal{I}(P)$ of its computation diagrammatically displayed as in Table 3. The on-line Szondi-test [14] also uses this program as a subroutine. Lines starting with “//” are

Listing 1: Update algorithm

```

1 public void update (Vector<Signature[]> profiles) {
2     // 1. CALCULATION OF MATERIAL IMPLICATIONS
3     // in the first profile
4     Signature[] fstp = profiles.firstElement();
5     // consequent-oriented processing (consequent loop),
6     // round-robin treatment of each factor as consequent
7     for (int c=0; c<8; c++) {
8         Signature cmodq = moduloQuanta(fstp[c]);
9         // 1.1 EVERYTHING IMPLIES TRUTH (antecedent loop),
10        // round-robin treatment of each factor as antecedent
11        for (int a=0; a<8; a++) {
12            // signature-value loop
13            for (int v=0; v<4; v++) {
14                // discount corresponding table-cell value
15                (factors[a][c]).signatures[v][code(cmodq)]--;
16            }
17        }
18        // 1.2 FALSEHOOD IMPLIES EVERYTHING—ALSO FALSEHOOD;
19        // everything is: all other consequent signatures
20        for (Signature cc : coSet(cmodq)) {
21            // round-robin treatment of each factor as antecedent
22            for (int a=0; a<8; a++) {
23                // false is: all other antecedent signatures
24                for (Signature ca : coSet(moduloQuanta(fstp[a]))) {
25                    // discount corresponding table-cell value
26                    (factors[a][c]).signatures[code(ca)][code(cc)]--;
27                }
28            }
29        }
30    }
31    // 2. CALCULATION OF INTUITIONISTIC IMPLICATIONS:
32    // forward invariance of material implications
33    if (profiles.size()>1) {
34        // garbage-collect the processed profile
35        profiles.remove(0);
36        // recursively descend on the remaining profiles
37        update(profiles);
38    }
39 }

```

Table 4: Material implication \supset

| | A | A' | $A \supset A'$ |
|----|-------|-------|----------------|
| 1. | false | false | true |
| 2. | false | true | true |
| 3. | true | false | false |
| 4. | true | true | true |

comments. Notice that every loop in the program has fixed complexity, and that we simply process the head of P —the first profile in P —in Line 2–30 and then recur on the tail of the remaining profiles in P in Line 31–38. The loop-nesting depth is four. The program updates a table—called *factors* in Listing 1—of eight times eight subtables—called *signatures* in Listing 1—of four times four content cells as displayed in Table 3, each of whose cells is initialised with a value equal to the length of P (e.g., 10). To update this table means to discount the initial value of its cells according to the following strategy inspired by Kripke’s model-theoretic interpretation of intuitionistic implication as forward invariance of material implication [15, 17] and adapted to our setting in Definition 3:

1. Calculate all material implications in the first profile in P , called *profiles* in Listing 1, according to the definition of material implication recalled in Table 4. There, Line 1, 2, and 4 can be summarised by the slogan “Everything implies truth” and Line 1 and 2 by the slogan “Falsehood implies everything.” In Listing 1, these well-known slogans correspond to the meaning of our code in Line 9–17 and Line 18–29, respectively. There, the function *moduloQuanta* simply returns its argument signature without quanta for graphical tractability, the function *code* the subtable line number of its argument, and the function *coSet* the set of all plain signatures (those signatures without quanta) minus the argument signature. For example, applying
 - *moduloQuanta* to the signature $+!!!$ returns the signature $+$,
 - *code* to the signature $+$ returns the line number 1, and
 - *coSet* to the signature $+$ returns the set of signatures $\{-, 0, \pm\}$.
2. Then calculate those material implications that are actually even intuitionistic implications by recurring on the tail of P .

On termination of program execution, each table cell that corresponds to an intuitionistic implication will contain the number 0 and be painted black. Cells containing the number 1 will be painted red (and correspond to intuitionistic implications of the tail of P), those containing the number 2 will be painted orange, and those containing the number 3 yellow. Table cells containing other numbers will not be painted for lack of relevance and thus will just display the number of missing discounts as distance to count as intuitionistic implications.

Observe in Table 3 that the whole diagonal from the top left corner down to the bottom right corner is painted black. This state of affairs reflects the reflexivity property $\vdash_{\Gamma} \phi \rightarrow \phi$ of intuitionistic implication. Similarly, if both a cell representing some formula $A \rightarrow A'$ as well as another cell representing some formula $A' \rightarrow A''$ are painted black then the cell representing the formula $A \rightarrow A''$ will also be painted black. Consider the following four example triples:

- $e+ \rightarrow s0$, $s0 \rightarrow k\pm$, and $e+ \rightarrow k\pm$;
- $k+ \rightarrow s-$, $s- \rightarrow p+$, and $k+ \rightarrow p+$;
- $p\pm \rightarrow s0$, $s0 \rightarrow k\pm$, and $p\pm \rightarrow k\pm$;
- $m\pm \rightarrow d+$, $d+ \rightarrow hy\pm$, and $m\pm \rightarrow hy\pm$.

This state of affairs reflects the transitivity property of intuitionistic implication, which for general formulas ϕ and ϕ' is:

$$\text{if } \vdash_{\Gamma} \phi \rightarrow \phi' \text{ and } \vdash_{\Gamma} \phi' \rightarrow \phi'' \text{ then } \vdash_{\Gamma} \phi \rightarrow \phi''.$$

Of course, reflexivity and transitivity are two logical properties, which will show up in the diagram of any $\mathcal{I}(P)$. (In so far, these properties also reflect an axiomatic redundancy of $\mathcal{I}(P)$, which however is not our concern here.)

In contrast, the following properties displayed in Table 3 are *psychological* in that they are proper to the personality profile displayed in Table 2, from which they have been extracted, namely:

Vacuous implications This class of intuitionistic implications is visually characterised by a horizontal or vertical line of black cells throughout the whole diagram width and diagram height, respectively. In the diagram displayed in Table 3, there is a single vertical line of such implications, which says:

Whenever something is true then $h-$ is true.

This simply holds because $h-$ is true throughout the whole test result in Table 2, and thus the logical slogan “Everything implies truth” applies.

The other, that is, the horizontal lines of vacuous intuitionistic implications hold due to the other logical slogan “Falsehood implies everything.”

Non-vacuous implications This class of intuitionistic implications is visually characterised by isolated black cells. They are the psychologically truly interesting implications. They are, from the top left to the bottom right of the diagram in Table 3, and together with their rough psychological meaning in Szondi’s system (recall Table 1 and, if need be, consult [21]):

1. $s0 \rightarrow k\pm$. Whenever the testee is inactive (externally) then he has internal compulsive behaviour (e.g., is experiencing a dilemma). See for example the below Item 3 and 6, where these two implicationally related reactions also appear conjunctively.

2. $s- \rightarrow p+$. Whenever the testee is receptively passive (e.g., masochism) then he wants to be more than he actually is (e.g., megalomania). See for example the below Item 5, where these two implicationally related reactions also appear conjunctively.
3. $e+ \rightarrow (s0 \wedge hy\pm \wedge k\pm \wedge p+ \wedge d0 \wedge m+)$: Whenever the testee has ethical behaviour then he
 - (a) is inactive ($s0$). That is, inactivity is a necessary condition for the testee's ethical behaviour, and thus his ethical behaviour is in a behavioural (not logical) sense vacuous.
 - (b) is morally ambivalent ($hy\pm$). Thus the testee's ethical behaviour need not be moral. Indeed, inactivity need not be moral.
 - (c) has internal compulsive behaviour ($k\pm$). Maybe the testee's inactivity is due to an internally experienced dilemma?
 - (d) wants to be more than he is ($p+$). The testee's inactivity may not be conducive to the fulfilment of his desire, but his desire may well be co-determined by his inactivity.
 - (e) is faithfully indifferent ($d0$). Indeed, faithfulness (in a general sense) and ethics may be experienced as orthogonal issues.
 - (f) approves of bindings in his relationships ($m+$). Thus for the testee, bindings but not necessarily their faithfulness are ethical.
4. $hy+ \rightarrow (d0 \wedge m+)$: Whenever the testee has immoral behaviour then he
 - (a) is faithfully indifferent ($d0$).
 - (b) approves of bindings in his relationships ($m+$).

Thus the testee's faithfulness indifference as well as his binding attitude is stable with respect to ethics and immorality.
5. $k+ \rightarrow (s- \wedge p+)$: Whenever the testee wants to have more than he has then he
 - (a) is receptively passive ($s-$).
 - (b) wants to be more than he is ($p+$).

Again, the testee's receptive passivity may not be conducive to the fulfilment of his desires, but his desires may well be co-determined by his receptive passivity.
6. $p\pm \rightarrow (s0 \wedge k\pm)$: Whenever the testee is ambivalent with respect to being more or less than he is then he
 - (a) is inactive ($s0$). The testee's ambivalence may well be a deeper dilemma that is the cause of his activity blockage.
 - (b) has internal compulsive behaviour ($k\pm$). This could be a confirmation of the testee's suspected dilemma.
7. $d0 \rightarrow m+$. Whenever the testee is faithfully indifferent then he approves of bindings in his relationships. See for example the above Item 3 and 4, where these two implicationally related reactions also appear conjunctively.

8. $d+ \rightarrow hy\pm$. Whenever the testee is unfaithful then he is morally ambivalent. See for example the below Item 10, where these two implicationally related reactions also appear conjunctively.
9. $m+ \rightarrow d0$. Whenever the testee approves of bindings in his relationships then he is faithfully indifferent. For example see Item 3 and 4, where these two implicationally related reactions also appear conjunctively.
10. $m\pm \rightarrow (hy\pm \wedge d+)$: Whenever the testee is ambivalent in his attitude towards bindings in his relationships then he
 - (a) is morally ambivalent ($hy\pm$).
 - (b) is unfaithful ($d+$).

Observe that from the above invariants in the given P we can deduce that:

$$\begin{aligned}
 &\vdash_{\mathcal{I}(P)} (e+ \vee s0 \vee p\pm) \rightarrow k\pm \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee s- \vee k+) \rightarrow p+ \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee hy+) \rightarrow (d0 \wedge m+) \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee hy+ \vee m+) \rightarrow d0 \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee hy+ \vee d0) \rightarrow m+ \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee p\pm) \rightarrow s0 \\
 &\vdash_{\mathcal{I}(P)} (e+ \vee d+ \vee m\pm) \rightarrow hy\pm
 \end{aligned}$$

From our diagrammatic reasonings, it becomes clear that the signed factor $e+$ and to a lesser extent the signed factor $hy+$ are the two most important *causal factors* in P —and thus for the testee represented by P —in the following sense:

1. these factors are non-vacuously implied by no other signed factor, but
2. they individually and non-vacuously imply most other signed factors.

Thus the testee's personality is determined to a large extent by these two signed factors, in spite of the fact that they only occur in P once and twice, respectively!

Diagrammatic reasoning in a couple Given two (or more) SPP-sequences P and P' , we can compute their axiom bases $\mathcal{I}(P)$ and $\mathcal{I}(P')$, and visualise them as diagrams. We can then graphically compute the join and the meet of P and P' , that is, the union and the intersection of $\mathcal{I}(P)$ and $\mathcal{I}(P')$, respectively, by simply superposing the two diagrams as printed on overhead-projector foils, and then adding their cells and pinpointing their common cells on a third and fourth superimposed foil, respectively. Of course, this graphical computation can instead also be programmed on a computer (e.g., for studying groups).

Conjunctive implicational invariants As indicated, our algorithmically extracted implicational invariants $A \rightarrow A'$ are simple in that they have a single

atomic antecedent A and a single atomic consequent A' . An interesting generalisation of these simple implicational invariants is to allow finite conjunctions $A_1 \wedge \dots \wedge A_n$ of atomic formulas $A_1, \dots, A_n \in \mathbb{A}$ as antecedents. This generalisation, though graphically not generally tractable in two dimensions, is interesting because with it, signed personality factors, represented by atomic formulas, can be analysed in terms of their *individually necessary and jointly sufficient conditions*, and thus *logically characterised in terms of each other*. More precisely, we mean by this characterisation that from

1. $\vdash_{\mathcal{I}(P)} (A_1 \wedge \dots \wedge A_n) \rightarrow A'$, that is, the atomic formulas A_1, \dots, A_n are *jointly sufficient conditions* for the atomic formula A' , and
2. $\vdash_{\mathcal{I}(P)} (A' \rightarrow A_1) \wedge \dots \wedge (A' \rightarrow A_n)$, that is, the atomic formulas A_1, \dots, A_n are *individually necessary conditions* for the atomic formula A' ,

we can deduce the following *equivalence characterisation* of A' :

$$\vdash_{\mathcal{I}(P)} (A_1 \wedge \dots \wedge A_n) \leftrightarrow A'.$$

Obviously, the truth of Item 2 can be ascertained graphically with our automatic procedure, and Item 1 can be ascertained interactively with the following semi-automatic procedure, involving a standard, efficient database query language:

1. Transcribe the given P (e.g., Table 2) into *non-recursive Datalog* [1];
2. Formulate and then query the resulting database with the jointly sufficient conditions that you suspect to be true.

Proposition 1 (Suffix closure of \mathcal{I}). *For every $P, P' \in \text{SPP}^+$,*

1. $\mathcal{I}(P \star P') \subseteq \mathcal{I}(P')$
2. $P \sqsubseteq P'$ implies $\mathcal{I}(P) \subseteq \mathcal{I}(P')$

Proof. For (1), consider:

- | | | |
|----|---|------------|
| 1. | 1. $P, P' \in \text{SPP}^+$ | hypothesis |
| 2. | 2. $\phi \in \mathcal{I}(P \star P')$ | hypothesis |
| 3. | 3. there are $A, A' \in \mathbb{A}$ such that $\phi = A \rightarrow A'$ and for every $P'' \in \text{SPP}^+$, $P \star P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ | 2 |
| 4. | 4. $\phi = A \rightarrow A'$ and for every $P'' \in \text{SPP}^+$, $P \star P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ | hypothesis |
| 5. | 5. $P'' \in \text{SPP}^+$ | hypothesis |
| 6. | 6. $P' \sqsubseteq P''$ | hypothesis |
| 7. | 7. $P \star P' \sqsubseteq P''$ | 6, Fact 1 |
| 8. | 8. $\text{p}(\pi_1(P'')) \vdash \phi$ | 4, 5, 7 |
| 9. | 9. $P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ | 6–8 |

10. for every $P'' \in \text{SPP}^+$, $P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ 5–9
11. $\phi = A \rightarrow A'$ and for every $P'' \in \text{SPP}^+$,
 $P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ 4, 10
12. there are $A, A' \in \mathbb{A}$ such that $\phi = A \rightarrow A'$ and
for every $P'' \in \text{SPP}^+$, $P' \sqsubseteq P''$ implies $\text{p}(\pi_1(P'')) \vdash \phi$ 11
13. $\phi \in \mathcal{I}(P')$ 12
14. $\phi \in \mathcal{I}(P')$ 3, 4–13
15. $\mathcal{I}(P \star P') \subseteq \mathcal{I}(P')$ 2–14
16. for every $P, P' \in \text{SPP}^+$, $\mathcal{I}(P \star P') \subseteq \mathcal{I}(P')$ 1–15.

For (2), consider:

1. $P, P' \in \text{SPP}^+$ hypothesis
2. $P \sqsubseteq P'$ hypothesis
3. $P = P'$ or there is $P'' \in \text{SPP}^+$ such that $P = P'' \star P'$ 2
4. $P = P'$ implies $\{P\}^\triangleleft \subseteq \{P'\}^\triangleleft$ equality law
5. there is $P'' \in \text{SPP}^+$ such that $P = P'' \star P'$ hypothesis
6. $P'' \in \text{SPP}^+$ and $P = P'' \star P'$ hypothesis
7. $\phi \in \mathcal{I}(P)$ hypothesis
8. $\phi \in \mathcal{I}(P'' \star P')$ 6, 7
9. $\mathcal{I}(P'' \star P') \subseteq \mathcal{I}(P')$ 1, Proposition 1.1
10. $\phi \in \mathcal{I}(P')$ 8, 9
11. $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 7–10
12. $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 5, 6–11
13. there is $P'' \in \text{SPP}^+$ such that $P = P'' \star P'$
implies $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 5–12
14. $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 3, 4, 13
15. $P \sqsubseteq P'$ implies $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 2–14
16. for every $P, P' \in \text{SPP}^+$, $P \sqsubseteq P'$ implies $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ 1–15.

□

Proposition 2.

1. $\vdash_{\mathcal{I}(P \star P')} \phi$ implies $\vdash_{\mathcal{I}(P')} \phi$
2. If $P \sqsubseteq P'$ and $\vdash_{\mathcal{I}(P)} \phi$ then $\vdash_{\mathcal{I}(P')} \phi$.

Proof. Combine Fact 2.2 with Proposition 1.1 and Proposition 1.2, respectively.

□

The following property means that our personality theories have the desired prime-filter property (see Proposition 3), as announced in the introduction.

Theorem 1 (Disjunction Property).

$$\text{If } \vdash_{\mathcal{I}(P)} \phi \vee \phi' \text{ then } \vdash_{\mathcal{I}(P)} \phi \text{ or } \vdash_{\mathcal{I}(P)} \phi'.$$

Proof. Our proof strategy is to adapt de Jongh's strategy in [5] to our simpler setting, thanks to which our proof reduces to Gödel's proof of the disjunction property of a basic intuitionistic theory [8] such as our $\text{Cl}(\emptyset)$: So suppose that $\vdash_{\mathcal{I}(P)} \phi \vee \phi'$. Adapting an observation from [5], we can assert that $\vdash_{\mathcal{I}(P)} \phi \vee \phi'$ if and only if $\vdash_{\emptyset} \bigwedge \mathcal{I}(P) \rightarrow (\phi \vee \phi')$. Thus $\vdash_{\emptyset} \bigwedge \mathcal{I}(P) \rightarrow (\phi \vee \phi')$. Hence $\vdash_{\emptyset} (\bigwedge \mathcal{I}(P) \rightarrow \phi) \vee (\bigwedge \mathcal{I}(P) \rightarrow \phi')$. Hence $\vdash_{\emptyset} (\bigwedge \mathcal{I}(P) \rightarrow \phi)$ or $\vdash_{\emptyset} (\bigwedge \mathcal{I}(P) \rightarrow \phi')$ by Gödel's proof. Hence $\vdash_{\mathcal{I}(P)} \phi$ or $\vdash_{\mathcal{I}(P)} \phi'$ again by de Jongh's observation. \square

3 Personality categorisation

In this section, we present the part of our framework for the mathematical categorisation of personality axioms into formal personality theories, as these axioms might have been discovered with the methodology presented in the previous section. As announced in the introduction, personality theories and personality-test data are related by a Galois-connection [4, Chapter 7]. We start with defining this connection and the two personality (powerset) spaces that it connects.

Definition 4 (Personality algebras). Let the mappings $\triangleright : 2^{\mathcal{L}(\mathbb{A})} \rightarrow 2^{\text{SPP}^+}$, called *right polarity*, and $\triangleleft : 2^{\text{SPP}^+} \rightarrow 2^{\mathcal{L}(\mathbb{A})}$, called *left polarity*, be such that

- $\Phi^{\triangleright} := \{ P \in \text{SPP}^+ \mid \text{for every } \phi \in \Phi, \vdash_{\mathcal{I}(P)} \phi \}$ and
- $\mathcal{P}^{\triangleleft} := \{ \phi \in \mathcal{L}(\mathbb{A}) \mid \text{for every } P \in \mathcal{P}, \vdash_{\mathcal{I}(P)} \phi \}$.

Further let $\equiv \subseteq 2^{\mathcal{L}(\mathbb{A})} \times 2^{\mathcal{L}(\mathbb{A})}$ and $\equiv \subseteq 2^{\text{SPP}^+} \times 2^{\text{SPP}^+}$ be their *kernels*, that is, for every $\Phi, \Phi' \in 2^{\mathcal{L}(\mathbb{A})}$, $\Phi \equiv \Phi'$ by definition if and only if $\Phi^{\triangleright} = \Phi'^{\triangleright}$ and for every $\mathcal{P}, \mathcal{P}' \in 2^{\text{SPP}^+}$, $\mathcal{P} \equiv \mathcal{P}'$ by definition if and only if $\mathcal{P}^{\triangleleft} = \mathcal{P}'^{\triangleleft}$, respectively.

Then, for each one of the two (inclusion-ordered, Boolean) powerset algebras

$$\langle 2^{\mathcal{L}(\mathbb{A})}, \emptyset, \cap, \cup, \mathcal{L}(\mathbb{A}), \bar{\cdot}, \subseteq \rangle \xrightleftharpoons[\triangleleft]{\triangleright} \langle 2^{\text{SPP}^+}, \emptyset, \cap, \cup, \text{SPP}^+, \bar{\cdot}, \subseteq \rangle,$$

define its (*ordered*) *quotient join semi-lattice with bottom* (and thus idempotent commutative monoid) modulo its kernel as in Table 5.

Note that our focus is on the powerset and not on the quotient algebras. The purpose of the quotient algebras is simply to indicate the maximally definable algebraic structure in our context. As a matter of fact, only the join- but not the meet-operation is well-defined in the quotient algebra (see Corollary 1).

Proposition 3 (Basic properties of personality theories).

Table 5: Quotient algebras

| Statements | Test results |
|---|--|
| $\top := [\mathcal{L}(\mathbb{A})]_{\equiv}$ | $\top := [\text{SPP}^+]_{\equiv}$ |
| $[\Phi]_{\equiv} \sqcup [\Phi']_{\equiv} := [\Phi \cup \Phi']_{\equiv}$ | $[\mathcal{P}]_{\equiv} \sqcup [\mathcal{P}']_{\equiv} := [\mathcal{P} \cup \mathcal{P}']_{\equiv}$ |
| $\perp := [\emptyset]_{\equiv}$ | $\perp := [\emptyset]_{\equiv}$ |
| $[\Phi]_{\equiv} \sqsubseteq [\Phi']_{\equiv} \text{ :iff } [\Phi]_{\equiv} \sqcup [\Phi']_{\equiv} = [\Phi']_{\equiv}$ | $[\mathcal{P}]_{\equiv} \sqsubseteq [\mathcal{P}']_{\equiv} \text{ :iff } [\mathcal{P}]_{\equiv} \sqcup [\mathcal{P}']_{\equiv} = [\mathcal{P}']_{\equiv}$ |

1. $\{P\}^{\triangleleft} = (\text{Cl} \circ \mathcal{I})(P)$ (generalisation to sets)
2. $P \sqsubseteq P'$ implies $\{P\}^{\triangleleft} \subseteq \{P'\}^{\triangleleft}$ (monotonicity)
3. prime filter properties:
 - (a) if $\phi \in \{P\}^{\triangleleft}$ and $\phi' \in \{P\}^{\triangleleft}$ then $\phi \wedge \phi' \in \{P\}^{\triangleleft}$ (and vice versa)
 - (b) if $\phi \in \{P\}^{\triangleleft}$ and $\phi' \in \mathcal{L}(\mathbb{A})$ and $\phi \vdash_{\mathcal{I}(P)} \phi'$ then $\phi' \in \{P\}^{\triangleleft}$
 - (c) if $\phi \vee \phi' \in \{P\}^{\triangleleft}$ then $\phi \in \{P\}^{\triangleleft}$ or $\phi' \in \{P\}^{\triangleleft}$ (and vice versa)
 ($\{P\}^{\triangleleft}$ is an intuitionistic theory.)
4. for every $\phi, \phi', \phi'' \in \mathcal{L}(\mathbb{A})$,

$$\text{if } \phi \vee \phi', \phi \vee \phi'' \in \bigcap_{P \in \mathcal{P}} \{P\}^{\triangleleft} \text{ then } \phi \vee (\phi' \wedge \phi'') \in \bigcap_{P \in \mathcal{P}} \{P\}^{\triangleleft}$$

($\bigcap_{P \in \mathcal{P}} \{P\}^{\triangleleft}$ is a distributive filter.)

Proof. For (1), consider that

$$\begin{aligned}
 \{P\}^{\triangleleft} &= \{\phi \in \mathcal{L}(\mathbb{A}) \mid \text{for every } P' \in \{P\}, \vdash_{\mathcal{I}(P')} \phi\} \\
 &= \{\phi \in \mathcal{L}(\mathbb{A}) \mid \vdash_{\mathcal{I}(P)} \phi\} \\
 &= \{\phi \in \mathcal{L}(\mathbb{A}) \mid \phi \in \text{Cl}(\mathcal{I}(P))\} \\
 &= \text{Cl}(\mathcal{I}(P)) \\
 &= (\text{Cl} \circ \mathcal{I})(P)
 \end{aligned}$$

For (2), suppose that $P \sqsubseteq P'$. Hence $\mathcal{I}(P) \subseteq \mathcal{I}(P')$ by Proposition 1.2. Hence $\text{Cl}(\mathcal{I}(P)) \subseteq \text{Cl}(\mathcal{I}(P'))$ by Fact 2.2. Thus $\{P\}^{\triangleleft} \subseteq \{P'\}^{\triangleleft}$ by (1). (3.a) follows from the fact that $(\phi \rightarrow (\phi' \rightarrow (\phi \wedge \phi'))) \in \text{Cl}(\emptyset)$ (and $((\phi \wedge \phi') \rightarrow \phi), ((\phi \wedge \phi') \rightarrow \phi') \in \text{Cl}(\emptyset)$), Fact 2.2, and (1). For (3.b), inspect definitions, and for (3.c), Theorem 1, definitions, and (1). For (4), consider (3) and recall that intersections of prime filters are distributive filters [4, Exercise 10.9]. \square

Now note the two macro-definitions $\triangleright^{\triangleleft} := \triangleright \circ \triangleleft$ and $\triangleleft^{\triangleright} := \triangleleft \circ \triangleright$ with \circ being function composition, as usual (from right to left, as usual too).

Lemma 1 (Some useful properties of \triangleright and \triangleleft).

1. if $\Phi \subseteq \Phi'$ then $\Phi'^{\triangleright} \subseteq \Phi^{\triangleright}$ (\triangleright is antitone)
2. if $\mathcal{P} \subseteq \mathcal{P}'$ then $\mathcal{P}'^{\triangleleft} \subseteq \mathcal{P}^{\triangleleft}$ (\triangleleft is antitone)
3. $\mathcal{P} \subseteq (\mathcal{P}^{\triangleleft})^{\triangleright}$ ($\triangleright^{\triangleleft}$ is extensive)
4. $\Phi \subseteq (\Phi^{\triangleright})^{\triangleleft}$ ($\triangleleft^{\triangleright}$ is extensive)
5. $((\mathcal{P}^{\triangleleft})^{\triangleright})^{\triangleleft} = \mathcal{P}^{\triangleleft}$
6. $((\Phi^{\triangleright})^{\triangleleft})^{\triangleright} = \Phi^{\triangleright}$
7. $((\mathcal{P}^{\triangleleft})^{\triangleright})^{\triangleleft} = (\mathcal{P}^{\triangleleft})^{\triangleright}$ ($\triangleright^{\triangleleft}$ is idempotent)
8. $((\Phi^{\triangleright})^{\triangleleft})^{\triangleright} = (\Phi^{\triangleright})^{\triangleleft}$ ($\triangleleft^{\triangleright}$ is idempotent)
9. if $\mathcal{P} \subseteq \mathcal{P}'$ then $(\mathcal{P}^{\triangleleft})^{\triangleright} \subseteq (\mathcal{P}'^{\triangleleft})^{\triangleright}$ ($\triangleright^{\triangleleft}$ is monotone)
10. if $\Phi \subseteq \Phi'$ then $(\Phi^{\triangleright})^{\triangleleft} \subseteq (\Phi'^{\triangleright})^{\triangleleft}$ ($\triangleleft^{\triangleright}$ is monotone)

Proof. For (1), suppose that $\Phi \subseteq \Phi'$. Further suppose that $P \in \Phi'^{\triangleright}$. That is, $\Phi' \subseteq \text{Cl}(\mathcal{I}(P))$. Now suppose that $\phi \in \Phi$. Hence $\phi \in \Phi'$. Hence $\phi \in \text{Cl}(\mathcal{I}(P))$. Thus $\Phi \subseteq \text{Cl}(\mathcal{I}(P))$. That is, $P \in \Phi^{\triangleright}$. Thus $\Phi'^{\triangleright} \subseteq \Phi^{\triangleright}$. For (2), suppose that $\mathcal{P} \subseteq \mathcal{P}'$. Further suppose that $\phi \in \mathcal{P}'^{\triangleleft}$. That is, for every $P \in \mathcal{P}'$, $\phi \in \text{Cl}(\mathcal{I}(P))$. Now suppose that $P \in \mathcal{P}$. Hence $P \in \mathcal{P}'$. Hence $\phi \in \text{Cl}(\mathcal{I}(P))$. Thus for every $P \in \mathcal{P}$, $\phi \in \text{Cl}(\mathcal{I}(P))$. That is, $\phi \in \mathcal{P}^{\triangleleft}$. Thus $\mathcal{P}'^{\triangleleft} \subseteq \mathcal{P}^{\triangleleft}$. For (3), suppose that $P \in \mathcal{P}$. Further suppose that $\phi \in \mathcal{P}^{\triangleleft}$. That is, for every $P \in \mathcal{P}$, $\phi \in \text{Cl}(\mathcal{I}(P))$. Hence $\phi \in \text{Cl}(\mathcal{I}(P))$. Thus $\mathcal{P}^{\triangleleft} \subseteq \text{Cl}(\mathcal{I}(P))$. That is, $P \in (\mathcal{P}^{\triangleleft})^{\triangleright}$. Thus $\mathcal{P} \subseteq (\mathcal{P}^{\triangleleft})^{\triangleright}$. For (4), suppose that $\phi \in \Phi$. Further suppose that $P \in \Phi^{\triangleright}$. That is, $\Phi \subseteq \text{Cl}(\mathcal{I}(P))$. Hence $\phi \in \text{Cl}(\mathcal{I}(P))$. Thus for every $P \in \Phi^{\triangleright}$, $\phi \in \text{Cl}(\mathcal{I}(P))$. That is, $\phi \in (\Phi^{\triangleright})^{\triangleleft}$. Thus $\Phi \subseteq (\Phi^{\triangleright})^{\triangleleft}$. For (5), consider that $\mathcal{P}^{\triangleleft} \subseteq ((\mathcal{P}^{\triangleleft})^{\triangleright})^{\triangleleft}$ is an instance of (4), and that $((\mathcal{P}^{\triangleleft})^{\triangleright})^{\triangleleft} \subseteq \mathcal{P}^{\triangleleft}$ by the application of (2) to (3). For (6), consider that $\Phi^{\triangleright} \subseteq ((\Phi^{\triangleright})^{\triangleleft})^{\triangleright}$ is an instance of (3), and that $((\Phi^{\triangleright})^{\triangleleft})^{\triangleright} \subseteq \Phi^{\triangleright}$ by the application of (1) to (4). For (7) and (8), substitute $\mathcal{P}^{\triangleleft}$ for Φ in (6), and Φ^{\triangleright} for \mathcal{P} in (5), respectively. For (9) and (10), transitively apply (1) to (2) and (2) to (1), respectively. \square

Notice that Lemma 1.3, 1.7, and 1.9 together with Lemma 1.4, 1.8, and 1.10 mean that $\triangleleft^{\triangleright}$ and $\triangleright^{\triangleleft}$ are closure operators, which is a fact relevant to Theorem 3.

Theorem 2 (The Galois-connection property of $(\triangleright, \triangleleft)$). *The ordered pair $(\triangleright, \triangleleft)$ is an antitone or order-reversing Galois-connection between the powerset algebras in Definition 4. That is, for every $\Phi \in 2^{\mathcal{L}(\mathbb{A})}$ and $\mathcal{P} \in 2^{\text{SPP}^+}$,*

$$\mathcal{P} \subseteq \Phi^{\triangleright} \text{ if and only if } \Phi \subseteq \mathcal{P}^{\triangleleft}.$$

Proof. Let $\Phi \in 2^{\mathcal{L}(\mathbb{A})}$ and $\mathcal{P} \in 2^{\text{SPP}^+}$ and suppose that $\mathcal{P} \subseteq \Phi^{\triangleright}$. Hence $(\Phi^{\triangleright})^{\triangleleft} \subseteq \mathcal{P}^{\triangleleft}$ by Lemma 1.2. Further, $\Phi \subseteq (\Phi^{\triangleright})^{\triangleleft}$ by Lemma 1.4. Hence $\Phi \subseteq \mathcal{P}^{\triangleleft}$ by transitivity. Conversely suppose that $\Phi \subseteq \mathcal{P}^{\triangleleft}$. Hence $(\mathcal{P}^{\triangleleft})^{\triangleright} \subseteq \Phi^{\triangleright}$ by Lemma 1.1. Further, $\mathcal{P} \subseteq (\mathcal{P}^{\triangleleft})^{\triangleright}$ by Lemma 1.3. Hence $\mathcal{P} \subseteq \Phi^{\triangleright}$. \square

Galois-connections are connected to *residuated mappings* [2].

Theorem 3 (De-Morgan like laws).

1. $(\mathcal{P} \cup \mathcal{P}')^\triangleleft = \mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft = ((\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft \subseteq \mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq ((\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$
2. $(\Phi \cup \Phi')^\triangleright = \Phi^\triangleright \cap \Phi'^\triangleright = ((\Phi^\triangleright \cap \Phi'^\triangleright)^\triangleleft)^\triangleright \subseteq \Phi^\triangleright \cup \Phi'^\triangleright \subseteq ((\Phi^\triangleright \cup \Phi'^\triangleright)^\triangleleft)^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$

Proof. For $(\mathcal{P} \cup \mathcal{P}')^\triangleleft = \mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft$ (join becomes meet) in (1), let $\phi \in \mathcal{L}(\mathbb{A})$, and consider that $\phi \in (\mathcal{P} \cup \mathcal{P}')^\triangleleft$ if and only if (for every $P \in \mathcal{P} \cup \mathcal{P}'$, $\phi \in \text{Cl}(\mathcal{I}(P))$) if and only if [for every P , ($P \in \mathcal{P}$ or $P \in \mathcal{P}'$) implies $\phi \in \text{Cl}(\mathcal{I}(P))$] if and only if [for every P , ($P \in \mathcal{P}$ implies $\phi \in \text{Cl}(\mathcal{I}(P))$) and ($P \in \mathcal{P}'$ implies $\phi \in \text{Cl}(\mathcal{I}(P))$)] if and only if [(for every $P \in \mathcal{P}$, $\phi \in \text{Cl}(\mathcal{I}(P))$) and (for every $P \in \mathcal{P}'$, $\phi \in \text{Cl}(\mathcal{I}(P))$)] if and only if ($\phi \in \mathcal{P}^\triangleleft$ and $\phi \in \mathcal{P}'^\triangleleft$) if and only if $\phi \in \mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft$. Then, $\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft \subseteq \mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft$ by elementary set theory. For later use of $\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ in (1) consider:

- | | | |
|-----|---|--------------------------|
| 1. | 1. $\phi \in \mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft$ | hypothesis |
| 2. | 2. $\phi \in \mathcal{P}^\triangleleft$ or $\phi \in \mathcal{P}'^\triangleleft$ | 1 |
| 3. | 3. $\phi \in \mathcal{P}^\triangleleft$ | hypothesis |
| 4. | 4. $P \in \mathcal{P} \cap \mathcal{P}'$ | hypothesis |
| 5. | 5. $P \in \mathcal{P}$ and $P \in \mathcal{P}'$ | 4 |
| 6. | 6. $P \in \mathcal{P}$ | 5 |
| 7. | 7. $\{P\} \subseteq \mathcal{P}$ | 6 |
| 8. | 8. $\mathcal{P}^\triangleleft \subseteq \{P\}^\triangleleft$ | 7, Lemma 1.2 |
| 9. | 9. $\phi \in \{P\}^\triangleleft$ | 3, 8 |
| 10. | 10. $\phi \in \text{Cl}(\mathcal{I}(P))$ | 9 |
| 11. | 11. for every $P \in \mathcal{P} \cap \mathcal{P}'$, $\phi \in \text{Cl}(\mathcal{I}(P))$ | 4–10 |
| 12. | 12. $\phi \in (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ | 11 |
| 13. | 13. if $\phi \in \mathcal{P}^\triangleleft$ then $\phi \in (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ | 3–12 |
| 14. | 14. if $\phi \in \mathcal{P}'^\triangleleft$ then $\phi \in (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ | similarly to 3–12 for 13 |
| 15. | 15. $\phi \in (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ | 2, 13, 14 |
| 16. | 16. $\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ | 1–15. |

For $((\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ in (1), consider the previously proved property that $\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$. Hence $(\mathcal{P} \cap \mathcal{P}') \subseteq (\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft)^\triangleright$ by Theorem 2. Hence $((\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$ by Lemma 1.2. Then, $\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq ((\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft$ in (1) is an instance of Lemma 1.4. For $\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft = ((\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft$ in (1), consider that $((\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft = (((\mathcal{P} \cup \mathcal{P}')^\triangleleft)^\triangleright)^\triangleleft$ by the previously proved property that $\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft = (\mathcal{P} \cap \mathcal{P}')^\triangleleft$. But $((\mathcal{P} \cup \mathcal{P}')^\triangleleft)^\triangleright = (\mathcal{P} \cup \mathcal{P}')^\triangleleft$ by Lemma 1.5. Hence $((\mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft)^\triangleright)^\triangleleft = \mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft$.

For $(\Phi \cup \Phi')^\triangleright = \Phi^\triangleright \cap \Phi'^\triangleright$ (join becomes meet) in (2), let $P \in \text{SPP}^+$, and consider that $P \in (\Phi \cup \Phi')^\triangleright$ if and only if (for every $\phi \in \Phi \cup \Phi'$, $\phi \in \text{Cl}(\mathcal{I}(P))$) if and only if [for every ϕ , ($\phi \in \Phi$ or $\phi \in \Phi'$) implies $\phi \in \text{Cl}(\mathcal{I}(P))$] if and only if [for

every ϕ , ($\phi \in \Phi$ implies $\phi \in \text{Cl}(\mathcal{I}(P))$) and ($\phi \in \Phi'$ implies $\phi \in \text{Cl}(\mathcal{I}(P))$) if and only if [(for every $\phi \in \Phi$, $\phi \in \text{Cl}(\mathcal{I}(P))$) and (for every $\phi \in \Phi'$, $\phi \in \text{Cl}(\mathcal{I}(P))$)] if and only if ($P \in \Phi^\triangleright$ and $P \in \Phi'^\triangleright$) if and only if $P \in \Phi^\triangleright \cap \Phi'^\triangleright$. Then, $\Phi^\triangleright \cap \Phi'^\triangleright \subseteq \Phi^\triangleright \cup \Phi'^\triangleright$ by elementary set theory. For later use of $\Phi^\triangleright \cup \Phi'^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$ in (2) consider:

| | | |
|-----|--|--------------------------|
| 1. | $P \in \Phi^\triangleright \cup \Phi'^\triangleright$ | hypothesis |
| 2. | $P \in \Phi^\triangleright$ or $P \in \Phi'^\triangleright$ | 1 |
| 3. | $P \in \Phi^\triangleright$ | hypothesis |
| 4. | $\phi \in \Phi \cap \Phi'$ | hypothesis |
| 5. | $\phi \in \Phi$ and $\phi \in \Phi'$ | 4 |
| 6. | $\phi \in \Phi$ | 5 |
| 7. | $\{\phi\} \subseteq \Phi$ | 6 |
| 8. | $\Phi^\triangleright \subseteq \{\phi\}^\triangleright$ | 7, Lemma 1.1 |
| 9. | $P \in \{\phi\}^\triangleright$ | 3, 8 |
| 10. | $\phi \in \text{Cl}(\mathcal{I}(P))$ | 9 |
| 11. | for every $\phi \in \Phi \cap \Phi'$, $\phi \in \text{Cl}(\mathcal{I}(P))$ | 4–10 |
| 12. | $P \in (\Phi \cap \Phi')^\triangleright$ | 11 |
| 13. | if $P \in \Phi^\triangleright$ then $P \in (\Phi \cap \Phi')^\triangleright$ | 3–12 |
| 14. | if $P \in \Phi'^\triangleright$ then $P \in (\Phi \cap \Phi')^\triangleright$ | similarly to 3–12 for 13 |
| 15. | $P \in (\Phi \cap \Phi')^\triangleright$ | 2, 13, 14 |
| 16. | $\Phi^\triangleright \cup \Phi'^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$ | 1–15. |

For $((\Phi^\triangleright \cup \Phi'^\triangleright)^\triangleleft)^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$ in (2), consider the previously proved property that $\Phi^\triangleright \cup \Phi'^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$. Hence $(\Phi \cap \Phi') \subseteq (\Phi^\triangleright \cup \Phi'^\triangleright)^\triangleleft$ by Theorem 2. Hence $((\Phi^\triangleright \cup \Phi'^\triangleright)^\triangleleft)^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$ by Lemma 1.1. Then, $\Phi^\triangleright \cup \Phi'^\triangleright \subseteq ((\Phi^\triangleright \cup \Phi'^\triangleright)^\triangleleft)^\triangleright$ in (2) is an instance of Lemma 1.3. For $\Phi^\triangleright \cap \Phi'^\triangleright = ((\Phi^\triangleright \cap \Phi'^\triangleright)^\triangleleft)^\triangleright$ in (2), consider that $((\Phi^\triangleright \cap \Phi'^\triangleright)^\triangleleft)^\triangleright = (((\Phi \cup \Phi')^\triangleright)^\triangleleft)^\triangleright$ by the previously proved property that $\Phi^\triangleright \cap \Phi'^\triangleright = (\Phi \cup \Phi')^\triangleright$. But $((\Phi \cup \Phi')^\triangleright)^\triangleleft = (\Phi \cup \Phi')^\triangleright$ by Lemma 1.6. Hence $((\Phi^\triangleright \cap \Phi'^\triangleright)^\triangleleft)^\triangleright = \Phi^\triangleright \cap \Phi'^\triangleright$. \square

Corollary 1. *The quotient algebras in Table 5 are well-defined, that is, the equivalence relations $\equiv \subseteq 2^{\mathcal{L}(\mathbb{A})} \times 2^{\mathcal{L}(\mathbb{A})}$ and $\equiv \subseteq 2^{\text{SPP}^+} \times 2^{\text{SPP}^+}$ are congruences:*

1. if $\Phi \equiv \Phi'$ and $\Phi'' \equiv \Phi'''$ then $\Phi \cup \Phi'' \equiv \Phi' \cup \Phi'''$;
2. if $\mathcal{P} \equiv \mathcal{P}'$ and $\mathcal{P}'' \equiv \mathcal{P}'''$ then $\mathcal{P} \cup \mathcal{P}'' \equiv \mathcal{P}' \cup \mathcal{P}'''$.

Proof. By the De-Morgan like laws $(\Phi \cup \Phi')^\triangleright = \Phi^\triangleright \cap \Phi'^\triangleright$ and $(\mathcal{P} \cup \mathcal{P}')^\triangleleft = \mathcal{P}^\triangleleft \cap \mathcal{P}'^\triangleleft$, respectively (see Theorem 3). \square

We are finally ready for defining our announced personality categories, and this by means of our previously-defined Galois-connection.

Definition 5 (Personality categories). Let $\mathcal{P} \in 2^{\text{SPP}^+}$ and $\Phi \in 2^{\mathcal{L}(\mathbb{A})}$, and let

- $\mathcal{T}_\Phi := \{ \tau : 2^{\text{SPP}^+} \rightarrow 2^{\text{SPP}^+} \mid \begin{array}{l} \text{for every } \mathcal{P} \in 2^{\text{SPP}^+} \text{ and } \phi \in \Phi, \\ \phi \in \mathcal{P}^\triangleleft \text{ implies } \phi \in \tau(\mathcal{P})^\triangleleft \end{array} \}$ and
- $\mathcal{T}_\mathcal{P} := \{ \tau : 2^{\mathcal{L}(\mathbb{A})} \rightarrow 2^{\mathcal{L}(\mathbb{A})} \mid \begin{array}{l} \text{for every } \Phi \in 2^{\mathcal{L}(\mathbb{A})} \text{ and } P \in \mathcal{P}, \\ P \in \Phi^\triangleright \text{ implies } P \in \tau(\Phi)^\triangleright \end{array} \}$.

Then, define the categories (monoids)

$$\mathbf{T}_\Phi := \langle \mathcal{T}_\Phi, \text{id}, \circ \rangle \text{ and } \mathbf{T}_\mathcal{P} := \langle \mathcal{T}_\mathcal{P}, \text{id}, \circ \rangle$$

of Φ - and \mathcal{P} -preserving transformations, respectively.

Proposition 4 (Antitonicity properties of personality categories).

1. $\Phi \subseteq \Phi'$ implies $\mathcal{T}_{\Phi'} \subseteq \mathcal{T}_\Phi$
2. $\mathcal{P} \subseteq \mathcal{P}'$ implies $\mathcal{T}_{\mathcal{P}'} \subseteq \mathcal{T}_\mathcal{P}$
3. $P \subseteq P'$ implies $\mathcal{T}_{\{P'\}^\triangleleft} \subseteq \mathcal{T}_{\{P\}^\triangleleft}$
4. $\mathcal{T}_{\Phi \cup \Phi'} \subseteq \mathcal{T}_\Phi \subseteq \mathcal{T}_{\Phi \cap \Phi'}$
5. $\mathcal{T}_{\mathcal{P} \cup \mathcal{P}'} \subseteq \mathcal{T}_\mathcal{P} \subseteq \mathcal{T}_{\mathcal{P} \cap \mathcal{P}'}$

Proof. (1) and (2) follow straightforwardly from their respective definition, and (4) and (5) from (1) and (2), respectively. (3) follows from Proposition 3.2 and (1) by transitivity. \square

Proposition 5 (Preservation properties of personality transformations).

1. $\tau \in \mathcal{T}_{\Phi^\triangleright}$ implies $\Phi^\triangleright \subseteq \tau(\Phi)^\triangleright$
2. $\tau \in \mathcal{T}_{\mathcal{P}^\triangleleft}$ implies $\mathcal{P}^\triangleleft \subseteq \tau(\mathcal{P})^\triangleleft$
3. $\tau \in \mathcal{T}_{(\Phi \cap \Phi')^\triangleright}$ implies $(\Phi^\triangleright \subseteq \tau(\Phi)^\triangleright \text{ and } \Phi'^\triangleright \subseteq \tau(\Phi')^\triangleright)$
4. $\tau \in \mathcal{T}_{(\mathcal{P} \cap \mathcal{P}')^\triangleleft}$ implies $(\mathcal{P}^\triangleleft \subseteq \tau(\mathcal{P})^\triangleleft \text{ and } \mathcal{P}'^\triangleleft \subseteq \tau(\mathcal{P}')^\triangleleft)$

Proof. (1) and (2) follow by expansion of definitions. For (3) suppose that $\tau \in \mathcal{T}_{(\Phi \cap \Phi')^\triangleright}$. But by Theorem 3.2, $\Phi^\triangleright \cup \Phi'^\triangleright \subseteq (\Phi \cap \Phi')^\triangleright$. Hence $\mathcal{T}_{(\Phi \cap \Phi')^\triangleright} \subseteq \mathcal{T}_{\Phi^\triangleright \cup \Phi'^\triangleright}$ by Proposition 4.1. Hence $\tau \in \mathcal{T}_{\Phi^\triangleright \cup \Phi'^\triangleright}$. Hence $\tau \in \mathcal{T}_{\Phi^\triangleright}$ and $\tau \in \mathcal{T}_{\Phi'^\triangleright}$ by Proposition 4.4. Hence $\Phi^\triangleright \subseteq \tau(\Phi)^\triangleright$ and $\Phi'^\triangleright \subseteq \tau(\Phi')^\triangleright$ by (1). For (4) suppose that $\tau \in \mathcal{T}_{(\mathcal{P} \cap \mathcal{P}')^\triangleleft}$. But by Theorem 3.1, $\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft \subseteq (\mathcal{P} \cap \mathcal{P}')^\triangleleft$. Hence $\mathcal{T}_{(\mathcal{P} \cap \mathcal{P}')^\triangleleft} \subseteq \mathcal{T}_{\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft}$ by Proposition 4.2. Hence $\tau \in \mathcal{T}_{\mathcal{P}^\triangleleft \cup \mathcal{P}'^\triangleleft}$. Hence $\tau \in \mathcal{T}_{\mathcal{P}^\triangleleft}$ and $\tau \in \mathcal{T}_{\mathcal{P}'^\triangleleft}$ by Proposition 4.5. Hence $\mathcal{P}^\triangleleft \subseteq \tau(\mathcal{P})^\triangleleft$ and $\mathcal{P}'^\triangleleft \subseteq \tau(\mathcal{P}')^\triangleleft$ by (2). \square

4 Conclusions

We have provided a formal framework for the computer-aided discovery and categorisation of personality axioms as summarised in the abstract of this paper. Our framework is meant as a contribution towards practicing psychological research with the methods of the exact sciences, for obvious ethical reasons. Psychology workers (psychologists, psychiatrists, etc.) can now apply our visual framework in their own field (of) studies in order to discover personality theories and categories of their own interest. Our hope is that these field studies will lead to a mathematical systematisation of the academic discipline of psychology in the area of test-based personality theories with the help of our framework.

As future work on our current *synchronic* data analytics approach, which infers *perfect* implicational correlations (between human reactions) at a given time point (within a Szondi personality profile, an SPP) from their invariance across time (within an SPP-sequence, a Szondi-test result), *approximate* implicational correlations can be studied and a *diachronic* data analytics approach can be taken. Actually, our implicational diagrams such as Table 3 already contain such approximate implicational correlations in the form of cell values greater than 0, which as explained on Page 10 indicate the distance to invariance and thus the approximation to the perfection in question. This notion of approximate implicational correlation can be understood and further studied as a notion of *fuzzy implication* [10]. Then, a diachronic approach would mine correlations between *different* time points, typically one or several past or present and one or several future, in order to *forecast and predict* future reactions of the person in question, such as can be done with *Bayesian inference* [18] and *time series* analysis and forecasting [19]. Actually, Table 2 is such a time series.

Last but not least, we mention the only piece of related work [3] that we are aware of. There, the author develops a framework similarly motivated by invariance as ours, but with quite different setup, outcomes, and results. The author’s setup on the invariants side is a set of relations over a finite domain closed under the Boolean operations, whereas our corresponding setup is an intuitionistic theory, a certain set of propositional formulas, as induced by a data sequence (as exemplified by one produced by a personality test). On the transformations side, the author’s setup is a system of injective total functions, whereas our corresponding setup is a system of total functions *tout court*.

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